J Math Chem (2015) 53:1257–1273
DOI 10.1007/s10910-015-0485-7

ORIGINAL PAPER

Bounds for Jaeger integrals

Árpád Baricz · Tibor K. Pogány · Saminathan Ponnusamy · Imre Rudas

Received: 9 November 2014 / Accepted: 20 February 2015 / Published online: 4 March 2015
© Springer International Publishing Switzerland 2015

Abstract Lower and upper bounds are deduced for some Jaeger integrals which involve the Bessel functions of the first and second kind. The upper bounds contain some elementary functions as well as incomplete gamma functions, while the lower bounds are expressed also in terms of incomplete gamma functions and are deduced via some known inequalities for Bessel functions of the first and second kinds.

The research of Á. Baricz was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The work of Tibor K. Pogány was supported by the Croatian Science Foundation under the Project No. 5435/2014.

Á. Baricz
Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca, Romania

Á. Baricz · T. K. Pogány (✉) · I. Rudas
Institute of Applied Mathematics, Óbuda University, Budapest 1034, Hungary
e-mail: poganj@pfri.hr

Á. Baricz
e-mail: bariczocsi@yahoo.com

I. Rudas
e-mail: rudas@uni-obuda.hu

T. K. Pogány
Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Croatia

S. Ponnusamy
Indian Statistical Institute, Chennai Centre, Society for Electronic Transactions and Security, MGR Knowledge City, CIT Campus, Taramani, Chennai 600113, India
e-mail: samy@iitm.ac.in

Keywords Jaeger function · Lower incomplete Gamma function · Upper incomplete Gamma function · Bessel functions of the first and second kind · Modified Bessel function of the second kind

Mathematics Subject Classification Primary 26D15 · Secondary 33B20 · 33C10

1 Introduction

Let J_ν and Y_ν be the Bessel functions of the first and second kind of the order ν, respectively, and consider the following Jaeger integrals

$$\int_0^\infty \frac{\mathrm{e}^{-xu^2}}{J_0^2(u)+Y_0^2(u)}\frac{\mathrm{d}u}{u} \quad \text{and} \quad \int_0^\infty \frac{\mathrm{e}^{-xu^2}}{J_1^2(u)+Y_1^2(u)}\frac{\mathrm{d}u}{u}.$$

These integrals belong to a family of improper integrals first investigated by Jaeger [1] which are related to axisymmetric diffusion in contemporary electrochemistry, specifically chronopotentiometry and chronoamperometry. The first integral is relevant in chronoamperometry, and describes the axisymmetric diffusive transport from a columnar cylindrical conductor in circumstances where a potential (in the conductor) is applied and the resulting temporal current monitored. See [2] for more details and further references. Recently, Phillips and Mahon [2] reconsidered these kind of integrals and presented a detailed analysis on the less known numerical techniques necessary to solve such integrals and they employed them to obtain numerical solutions over a broad range of the temporal parameter x. Having in mind the importance of the Jaeger integrals and also by some authors' recent articles in which lower and upper bounds are established for certain special functions appearing in diverse applications like e.g. Coulomb function [3] and Bickley function [4], in this paper our aim is to make a contribution to the subject by considering a common generalization of the above two Jaeger integrals and to present some lower and upper bounds for this generalization. For $\nu > -1$ let us define the next generalized Jaeger integral of order ν, which is the following Laplace-form integral

$$G_\nu(x) = \int_0^\infty \eta_\nu(u)\,\mathrm{e}^{-xu^2}\,\mathrm{d}u, \qquad \eta_\nu(u) = \frac{1}{u(J_\nu^2(u)+Y_\nu^2(u))}.$$

In fact, to consider $G_\nu(x)$ we decided by the following approach to the heat flow mathematical model in an infinite solid bounded internally by a circular cylinder, see among others [5–7]. Considering the cylinder of radius $r = a$ whose boundary temperature V is raised at time $t = 0$, where the temperature field $v = v(r, \theta, z; t)$ behaves outside of the cylinder by linear heat equation with the diffusivity $\mathfrak{K}$:

$$\frac{\partial v}{\partial t} = \mathfrak{K}\,\nabla^2 v.$$

In solving this classical problem we apply the Laplace transform method to the subsidiary equation [7, p. 335], which reads

$$\frac{\mathrm{d}^2\overline{v}}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}\overline{v}}{\mathrm{d}r} - q^2\,\overline{v} = 0, \qquad r > 0, \tag{1.1}$$

where $\overline{v} = \overline{v}(p) = \int_0^\infty \mathrm{e}^{-pt}\,v(t)\,\mathrm{d}t$ stands for the Laplace transform of the temperature field v, and $q^2 = p\,\mathfrak{K}^{-1}$, under constraining that $\overline{v}$ is finite under $r \to \infty$ and $\overline{v}(a) = V\,p^{-1}$. The solution

$$\overline{v} = \frac{V\,K_0(qr)}{p\,K_0(qa)}$$

belongs to various authors, among others to Nicholson [8], Smith [5] and Carlslaw and Jaeger [7, p. 335, Eq. (3)]; here K_0 denotes the modified Bessel function of the second kind of zeroth order. By the Laplace inversion theorem we have [7, p. 335, Eq. (6)]

$$v = V + \frac{V}{\pi}\int_0^\infty \mathrm{e}^{-\mathfrak{K}u^2 t}\,\frac{J_0(ur)Y_0(ua) - J_0(ua)Y_0(ur)}{J_0^2(ua) + Y_0^2(ua)}\,\frac{\mathrm{d}u}{u}; \tag{1.2}$$

the related flux at the surface becomes [7, p. 336, Eq. (8)]

$$\phi = -\mathsf{C}\,\frac{\partial v}{\partial r}\bigg|_{r=a} = \frac{4\mathsf{C}V}{a\pi^2}\,G_0(\mathfrak{K}t).$$

Let us consider the modified Bessel type differential equation (1.1), rewritten into the equivalent form

$$r^2\overline{v}'' + r\,\overline{v}' - [\nu^2 + (qr)^2]\,\overline{v} = -\nu^2\,\overline{v} =: f_\nu(r) \tag{1.3}$$

is in turn a non-homogeneous modified Bessel differential equation. Its exact solution has been given by Chessin [9,10] for general f_ν; Chessin's method one reduces to the variation of parameters (see [11, p. 532, Theorem 2]). Accordingly, the solution of the homogeneous part is

$$\overline{v}_h = c_1 I_\nu(qr) + c_2 K_\nu(qr),$$

where I_ν, K_ν denote the modified Bessel functions of the first and the second kind, both of the order ν. It is known that I_ν, K_ν are independent solutions of the homogeneous Bessel DE, since the Wronskian $W[J(x), Y](x) = -x^{-1} < 0$ for positive arguments. A guess of the particular solution is $\overline{v}_p(r) = c_2 K_\nu(qr)w(r)$, because the choice $c_1 = 0$ is obviously legitimate. Substituting this into (1.3), we have

$$\begin{aligned}&\big\{(qr)^2K_\nu''(qr) + (qr)K_\nu'(qr) - [\nu^2 + (qr)^2]K_\nu(qr)\big\}w \\ &\quad + r[2qrK_\nu'(qr) + K_\nu(qr)]\,w' + K_\nu(qr)w'' = -\nu^2\,K_\nu(qr)w,\end{aligned}$$

where the first left-hand-side addend vanishes since K_ν is a solution of the homogeneous part of DE (1.3). Accordingly

$$w'' + \left(2q\,\frac{K_\nu'(qr)}{K_\nu(qr)} + \frac{1}{r}\right) w' + \frac{\nu^2}{r^2}\, w = 0. \tag{1.4}$$

Bearing in mind the boundary constraints, we conclude

$$\overline{v} = \frac{V\, K_\nu(qr)\, w(qr)}{p\, K_\nu(ar)\, w(ar)},$$

where w is the solution of the second order linear ODE (1.4). Now, by virtue of the similar Laplace inversion procedure that has been developed by Carlslaw and Jaeger [7, p. 334 *et seq.* §13.5] we conclude that the temperture field's description one transforms into the one expressible, instead of J_0, Y_0, *via* J_ν, Y_ν. Moreover, the flux ϕ is related to the extended Jaeger function G_ν. Therefore the thermo-, and electrochemical relevance of G_ν, concerning the heat conduction models in circular cylinder shaped solids is evident.

In the sequel our aim is to deduce some lower and upper bounds for the above generalized Jaeger integral. The upper bounds (deduced in Sect. 2) contain some elementary functions as well as incomplete gamma functions, while the lower bounds (presented in Sect. 3) are expressed also in terms of incomplete gamma functions and are deduced via some known inequalities for Bessel functions of the first and second kinds. These bounds may be useful in the further study of Jaeger integrals and can have applications in electrochemistry.

2 Upper bounds for Jaeger integrals

Our first main result concerns the relation between $G_\nu(x)$ and $G_{\frac{1}{2}}(x)$.

Theorem 1 *If* $\nu \in \left[0, \frac{1}{2}\right]$ *and* $x > 0$ *we have the following inequality*

$$G_\nu(x) \geq G_{\frac{1}{2}}(x) = \frac{\pi^{3/2}}{4\sqrt{x}}.$$

Moreover, for $\nu > \frac{1}{2}$ *and* $x > 0$ *the above inequality is reversed, that is, we have*

$$G_\nu(x) < \frac{\pi^{3/2}}{4\sqrt{x}}.$$

Proof Let us begin with calculating the value $G_{\frac{1}{2}}(x)$. The expression $\eta_{\frac{1}{2}}^{-1}(u)$ one reduces as

$$\eta_{\frac{1}{2}}^{-1}(u) = u\left(J_{\frac{1}{2}}^2(u) + Y_{\frac{1}{2}}^2(u)\right) = u\left(\frac{2}{\pi u}\sin^2 u + \frac{2}{\pi u}\cos^2 u\right) = \frac{2}{\pi},$$

and consequently

$$G_{\frac{1}{2}}(x) = \frac{\pi}{2}\int_0^\infty \mathrm{e}^{-xu^2}\mathrm{d}u = \frac{\pi}{2\sqrt{x}}\int_0^\infty \mathrm{e}^{-v^2}\mathrm{d}v = \frac{\pi^{3/2}}{4\sqrt{x}}.$$

It is known that $u \mapsto (\eta_\nu(u))^{-1}, u \in \mathbb{R}_+$ monotone increases for $\nu \in [0, \frac{1}{2})$, and decreases when $\nu > \frac{1}{2}$. Accordingly on the same u-domain

$$\eta_\nu(u) > \frac{\pi}{2}, \quad \nu \in \left[0, \tfrac{1}{2}\right); \qquad \text{and} \qquad \eta_\nu(u) < \frac{\pi}{2}, \quad \nu \in \left(\tfrac{1}{2}, \infty\right).$$

Now, obvious transformations lead to the assertions of the theorem. □

It is worth to mention that for $\nu = n + \frac{1}{2}, n \in \mathbb{N}$ there holds [12,13]

$$J^2_{n+\frac{1}{2}}(u) + Y^2_{n+\frac{1}{2}}(u) = \frac{2}{\pi}\, u^{-2n-1}\prod_{j=1}^{n}\left(u^2 + \alpha_j^2\right),$$

where $\alpha_j, j \in \{1, 2, \ldots, n\}$ are the consecutive zeros of the modified Bessel functions of the second kind $K_{n+\frac{1}{2}}$. We know that (see for example [14]) $K_{n+\frac{1}{2}}$ for $n \in \mathbb{N}$ has exactly n zeros with multiplicity one in $\{z \in \mathbb{C}\colon\ \mathrm{Re}\, z < 0\}$ and that the non-real zeros are complex conjugate in pairs, see [15]. Therefore we have

$$G_{n+\frac{1}{2}}(x) = \frac{\pi}{2}\int_0^\infty \frac{u^{2n}\,\mathrm{e}^{-xu^2}}{\prod_{j=1}^n\left(u^2 + \alpha_j^2\right)}\,\mathrm{d}u, \tag{2.1}$$

which is representable in the form of the sum of proper fractions

$$G_{n+\frac{1}{2}}(x) = \frac{\pi}{2}\sum_{j=1}^n B_j\int_0^\infty \frac{\mathrm{e}^{-xu^2}}{u^2+\alpha_j^2}\,\mathrm{d}u, \quad \text{where} \quad B_j = \frac{(-1)^n\,\alpha_j^{2n}}{\prod_{1\le k\le n,\ j\ne k}\left(\alpha_k^2 - \alpha_j^2\right)}.$$

Thus, for all $\mathrm{Re}\, a \neq 0$ and $\mathrm{Re}\, x > 0$ there holds

$$\int_0^\infty \frac{\mathrm{e}^{-xu^2}}{u^2+a^2}\,\mathrm{d}u = \frac{\pi}{2a}\,\mathrm{e}^{a^2x}\left(\frac{a}{|a|} - 1 + \mathrm{erfc}(a\sqrt{x})\right),$$

which implies that

$$G_{n+\frac{1}{2}}(x) = \frac{(-1)^n\pi^2}{4}\sum_{k=1}^n\sum_{j=1}^n \frac{\alpha_j^{2n-1}\,\mathrm{e}^{\alpha_j^2x}}{\prod_{\substack{1\le k\le n\\ j\ne k}}\left(\alpha_k^2 - \alpha_j^2\right)}\left(\frac{\alpha_j}{|\alpha_j|} - 1 + \mathrm{erfc}(\alpha_j\sqrt{x})\right),$$

where $u \mapsto \mathrm{erfc}(u)$ stands for the *complementary error-function*.

The second main result complements naturally Theorem 1.

Theorem 2 *If $n \in \mathbb{N}$ and $x > 0$, then we have that*

$$G_{n+\frac{1}{2}}(x) \leq \frac{\pi^{3/2}}{2^{n+2} x^{n+\frac{1}{2}} (2n-1)!!}.$$

Proof By using the integral representation (2.1) we have

$$\begin{aligned} G_{n+\frac{1}{2}}(x) &\leq \frac{\pi}{2 \prod_{j=1}^{n} \alpha_j^2} \int_0^\infty u^{2n} \, \mathrm{e}^{-xu^2} \, \mathrm{d}u \\ &= \frac{\pi}{4\, x^{n+\frac{1}{2}} \prod_{j=1}^{n} \alpha_j^2} \int_0^\infty \left(xu^2\right)^{n-\frac{1}{2}} \mathrm{e}^{-xu^2} \, \mathrm{d}\left(xu^2\right) \\ &= \frac{\pi}{4\, x^{n+\frac{1}{2}} \prod_{j=1}^{n} \alpha_j^2} \int_0^\infty v^{n-\frac{1}{2}} \, \mathrm{e}^{-v} \, \mathrm{d}v = \frac{\pi\, \Gamma\left(n+\frac{1}{2}\right)}{4\, x^{n+\frac{1}{2}} \prod_{j=1}^{n} \alpha_j^2}. \end{aligned}$$

On the other hand, from (2.1), we have

$$h_n(u) = \frac{\pi}{2} u^{2n+1} \left(J_{n+\frac{1}{2}}^2(u) + Y_{n+\frac{1}{2}}^2(u)\right) = \prod_{j=1}^{n} \left(u^2 + \alpha_j^2\right).$$

Since the zeros α_j, $j \in \{1, 2, \ldots\}$, occur in complex-conjugate pairs, we have that $h_n(0) = \prod_{j=1}^{n} \alpha_j^2 \in \mathbb{R}_+$. Moreover, the asymptotic expansion

$$h_n(u) = \frac{2^{2n}\, \Gamma^2\left(n+\frac{1}{2}\right)}{\pi} + o(u^2), \qquad u \to 0$$

confirms that

$$\prod_{j=1}^{n} \alpha_j^2 = \frac{2^{2n}\, \Gamma^2\left(n+\frac{1}{2}\right)}{\pi} = ((2n-1)!!)^2. \tag{2.2}$$

This completes the proof. □

Now, let us consider the *upper incomplete gamma function* by means of the the integral

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \, \mathrm{e}^{-t} \, \mathrm{d}t, \qquad |\arg(a)| < \pi,$$

which is an analytic function of both a and x, and it is entire for fixed values of $x > 0$. Simultaneously, we consider the *lower incomplete gamma function*

$$\gamma(a, x) = \Gamma(a) - \Gamma(a, x), \qquad \mathrm{Re}\, a > 0.$$

We will use also the terminology of the *generalized incomplete gamma function* $\Gamma(a, x_1, x_2) = \Gamma(a, x_2) - \Gamma(a, x_1)$. The next result provides a new upper bound for Jaeger integrals in terms of incomplete gamma functions.

Theorem 3 *Let* $m_\alpha = \min_{1\le j\le n} |\alpha_j|,\ M_\alpha = \max_{1\le j\le n} |\alpha_j|$ *and* $\kappa = k_1 + \cdots + k_n$. *Then for all* $n \in \mathbb{N}$ *and* $x > 0$ *we have*

$$G_{n+\frac{1}{2}}(x) \le \frac{\pi}{4} \left\{ \frac{1}{x^{n+\frac{1}{2}}\ [(2n-1)!!]^2} \prod_{j=1}^n \sum_{k_j\ge 0} \frac{\gamma\left(n+\kappa+\frac{1}{2}, xM_\alpha^2\right)}{\left(x|\alpha_j|^2\right)^{k_j}} + \frac{1}{x^{n+\frac{1}{2}}} \prod_{j=1}^n \sum_{k_j\ge 0} \left(\frac{|\alpha_j|^2}{x}\right)^{k_j} \left|\Gamma\left(n-\kappa+\frac{1}{2}, xm_\alpha^2\right)\right| + \frac{\Gamma\left(n+\frac{1}{2}, xm_\alpha^2, xM_\alpha^2\right)}{\prod_{j=1}^n \left(m_\alpha^2+\alpha_j^2\right)} \right\}.$$

Proof Consider the integral representation (2.1) in which transforming the integration domain we get

$$G_{n+\frac{1}{2}}(x) = \frac{\pi}{2} \left\{ \int_0^{M_\alpha} + \int_{m_\alpha}^{\infty} - \int_{m_\alpha}^{M_\alpha} \right\} \frac{u^{2n}\, \mathrm{e}^{-xu^2}}{\prod_{j=1}^n \left(u^2+\alpha_j^2\right)}\, \mathrm{d}u.$$

The first integral we can write as

$$\begin{aligned} I_1 &= \int_0^{M_\alpha} \frac{u^{2n}\, \mathrm{e}^{-xu^2}}{\prod_{j=1}^n \left(u^2+\alpha_j^2\right)}\, \mathrm{d}u = \frac{1}{\prod_{j=1}^n \alpha_j^2} \int_0^{M_\alpha} \frac{u^{2n}\, \mathrm{e}^{-xu^2}}{\prod_{j=1}^n \left(1+\dfrac{u^2}{\alpha_j^2}\right)}\, \mathrm{d}u \\ &= \frac{1}{[(2n-1)!!]^2} \prod_{j=1}^n \sum_{k_j\ge 0} \frac{(-1)^{k_j}}{\alpha_j^{2k_j}} \int_0^{M_\alpha} u^{2(n+\kappa)}\, \mathrm{e}^{-xu^2}\, \mathrm{d}u \\ &= \frac{1}{2x^{n+\frac{1}{2}}\ [(2n-1)!!]^2} \prod_{j=1}^n \sum_{k_j\ge 0} \left(-\frac{1}{x\alpha_j^2}\right)^{k_j} \int_0^{xM_\alpha^2} v^{n+\kappa-\frac{1}{2}}\, \mathrm{e}^{-v}\, \mathrm{d}v \\ &= \frac{1}{2x^{n+\frac{1}{2}}\ [(2n-1)!!]^2} \prod_{j=1}^n \sum_{k_j\ge 0} \left(-\frac{1}{x\alpha_j^2}\right)^{k_j} \gamma\left(n+\kappa+\tfrac{1}{2}, xM_\alpha^2\right), \end{aligned}$$

where $\kappa = k_1 + \cdots + k_n$. Immediately follows that

$$|I_1| \le \frac{1}{2x^{n+\frac{1}{2}}\ [(2n-1)!!]^2} \prod_{j=1}^n \sum_{k_j\ge 0} \frac{\gamma\left(n+\kappa+\frac{1}{2}, xM_\alpha^2\right)}{\left(x|\alpha_j|^2\right)^{k_j}}. \tag{2.3}$$

Similarly, the second integral can be written as

$$I_2 = \int_{m_\alpha}^{\infty} \frac{u^{2n}\, \mathrm{e}^{-xu^2}}{\prod_{j=1}^n \left(u^2+\alpha_j^2\right)}\, \mathrm{d}u = \frac{1}{2x^{n+\frac{1}{2}}} \prod_{j=1}^n \sum_{k_j\ge 0} \left(-\frac{\alpha_j^2}{x}\right)^{k_j} \int_{xm_\alpha^2}^{\infty} v^{n-\kappa-\frac{1}{2}}\, \mathrm{e}^{-v}\, \mathrm{d}v$$

$$= \frac{1}{2x^{n+\frac{1}{2}}} \prod_{j=1}^{n} \sum_{k_j \geq 0} \left(-\frac{\alpha_j^2}{x} \right)^{k_j} \Gamma\left(n - \kappa + \tfrac{1}{2}, xm_\alpha^2\right).$$

This representation implies the bound

$$|I_2| \leq \frac{1}{2x^{n+\frac{1}{2}}} \prod_{j=1}^{n} \sum_{k_j \geq 0} \left(\frac{|\alpha_j|^2}{x} \right)^{k_j} \left| \Gamma\left(n - \kappa + \tfrac{1}{2}, xm_\alpha^2\right) \right|. \tag{2.4}$$

Finally, for the third integral observe that since $m_\alpha \leq u \leq M_\alpha$ and the real polynomial $h_n(u)$, $u \in \mathbb{R}_+$ increases, we have

$$\min_{m_\alpha \leq u \leq M_\alpha} |h_n(u)| = h_n(m_\alpha) = \prod_{j=1}^{n} \left(m_\alpha^2 + \alpha_j^2 \right).$$

Thus, we have

$$\begin{aligned} I_3 &= \int_{m_\alpha}^{M_\alpha} \frac{u^{2n}\, \mathrm{e}^{-xu^2}}{\prod_{j=1}^{n} \left(u^2 + \alpha_j^2\right)} \mathrm{d}u \leq \frac{1}{2h_n(m_\alpha)} \int_{xm_\alpha^2}^{xM_\alpha^2} v^{n-\frac{1}{2}}\, \mathrm{e}^{-v} \mathrm{d}v \\ &= \frac{\Gamma\left(n + \frac{1}{2}, xm_\alpha^2, xM_\alpha^2\right)}{2h_n(m_\alpha)} \end{aligned} \tag{2.5}$$

and collecting the estimates (2.3), (2.4) and (2.5) by virtue of

$$G_{n+\frac{1}{2}}(x) = \left| G_{n+\frac{1}{2}}(x) \right| \leq \frac{\pi}{2} \left(|I_1| + |I_2| + |I_3| \right)$$

we arrive at the desired upper bound assertion. □

3 Lower bounds for Jaeger integrals

After establishing bounds for the Jaeger integral $G_\nu(x)$ either for real $\nu > 0$ or for more specialized values $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}$ we will made an attempt to derive certain general real ν order *lower bounds* for $G_\nu(x)$. For this purpose we list several bounding inequalities for $J_\nu(u)$ and $Y_\nu(u)$ on the positive real half-axis. Firstly, we mention von Lommel's [16, pp. 548–549] (see also [17, p. 406]) results

$$|J_\nu(u)| \leq 1, \quad |J_{\nu+1}(u)| \leq \frac{1}{\sqrt{2}}, \qquad \nu > 0,\ u \in \mathbb{R}, \tag{3.1}$$

and the bound by Minakshisundaram and Szász [18, p. 37]

$$|J_\nu(u)| \leq \frac{1}{\Gamma(\nu + 1)} \left(\frac{|u|}{2} \right)^\nu, \qquad u \in \mathbb{R}. \tag{3.2}$$

Another bounds were derived by Landau [19], who gave in a sense best possible bounds for the first kind Bessel function J_ν with respect to ν and u. These bounds read as follows

$$|J_\nu(u)| \le b_L\, \nu^{-1/3}, \qquad b_L = \sqrt[3]{2} \sup_{t\in\mathbb{R}_+} \mathrm{Ai}(t), \tag{3.3}$$

$$|J_\nu(u)| \le c_L\, |u|^{-1/3}, \qquad c_L = \sup_{t\in\mathbb{R}_+} t^{1/3} J_0(t), \tag{3.4}$$

where $\mathrm{Ai}(\cdot)$ stands for the Airy function. Note that Krasikov remarked that these bounds are sharp only in the transition region, i.e. for the values u around $j_{\nu,1}$, the first positive zero of $J_\nu(u)$ (see also Krasikov's related results [20]). It is also worth to mention that Olenko [21, Theorem 1] established the upper bound

$$\sup_{u\ge 0} \sqrt{u}|J_\nu(u)| \le b_L \sqrt{\nu^{1/3} + \frac{\alpha_1}{\nu^{1/3}} + \frac{3\alpha_1^2}{10\nu}} := d_O, \qquad \nu > 0, \tag{3.5}$$

where α_1 is the smallest positive zero of the Airy-function Ai and b_L is the Landau's constant above. Finally, among others, we mention the bound by Ifantis and Siafarikas [22, p. 215]

$$\big|J_\nu(u)\big| \le \frac{u^\nu}{2^\nu \Gamma(\nu+1)} \exp\left\{-\frac{u^2}{4(\nu+1)}\right\}, \qquad u > 0,\ \nu \ge 0. \tag{3.6}$$

We point out that Watson [17, p. 16] referred to Cauchy a specific variant of this bound (the exponential term contains $-u^2/4$), for integer order ν. For another kind upper bounds see for instance [22–25] and the related references therein.

Moreover, according to the best knowledge of the authors the only bound for the Bessel function of the second kind Y_ν belongs to Baricz et al. [26, pp. 957–958]; this estimate reads as follows

$$Y_\nu(u) \le \Upsilon_\nu(u)$$
$$= \begin{cases} \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{u}{2}\right)^\nu + \dfrac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac12\right)}\left(\dfrac{u}{2}\right)^{\nu-1}, & \nu \in \left(-\frac12, \frac12\right] \\ \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{u}{2}\right)^\nu + \dfrac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac12\right)}\left(\dfrac{u}{2}\right)^{\nu-1} + \dfrac{\Gamma(\nu)}{\pi}\left(\dfrac{2}{u}\right)^\nu, & \nu \in \left(\frac12, \frac32\right] \\ \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{u}{2}\right)^\nu + \dfrac{u^{\nu-1}}{\sqrt{2\pi}\,\Gamma\left(\nu+\frac12\right)} + \dfrac{2^{2\nu-\frac32}\Gamma(\nu)}{\pi u^\nu}, & \nu \in \left(\frac32, \infty\right) \end{cases}, \tag{3.7}$$

which holds for all positive u.

The forthcoming auxiliary lemmata follow by use of certain standard inequalities and the subsequent use of upper bounds (3.1)–(3.6) for J_ν which are twofold, uniform in u like von Lommel's (3.1) and Landau's (3.3) and the remaining non-uniform ones in u from the one side denoted by $\Xi_\nu(u)$, and the bound $\Upsilon_\nu(u)$ in (3.7) for Y_ν.

Now, we list two definite integral evaluation results.

Lemma 1 *For all* $\mu > 0,\ a > 0$ *and* $x > 0$ *there holds*

$$\mathscr{A}(a,\mu;x) = \int_0^\infty \frac{t^{\mu-1}\,\mathrm{e}^{-xt}}{a+t}\,\mathrm{d}t = a^{\mu-1}\mathrm{e}^{ax}\,\Gamma(\mu)\,\Gamma(1-\mu,ax). \tag{3.8}$$

Proof By using the known gamma-function property

$$\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-xt}\,\mathrm{d}t = \frac{\Gamma(\mu)}{x^\mu}, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(\mu) > 0,$$

we get by direct calculation, in which the legitimate integration order has been taken, that

$$\begin{aligned}\mathscr{A}(a,\mu;x) &= \int_0^\infty t^{\mu-1}\,\mathrm{e}^{-xt}\left\{\int_0^\infty \mathrm{e}^{-(a+t)s}\,\mathrm{d}s\right\}\mathrm{d}t\\ &= \int_0^\infty \mathrm{e}^{-as}\left\{\int_0^\infty t^{\mu-1}\,\mathrm{e}^{-(x+s)t}\,\mathrm{d}t\right\}\mathrm{d}s\\ &= \Gamma(\mu)\int_0^\infty \frac{\mathrm{e}^{-as}}{(x+s)^\mu}\,\mathrm{d}s = a^{\mu-1}\mathrm{e}^{ax}\,\Gamma(\mu)\int_{ax}^\infty q^{-\mu}\,\mathrm{e}^{-q}\,\mathrm{d}q,\end{aligned}$$

which is actually (3.8). □

Remark 1 It is worth to mention that the formula (3.8) can also be deduced by using the relation [27, p. 177, Eq. **8.6.4.**]:

$$\Gamma(p,z) = \frac{z^p\,\mathrm{e}^{-z}}{\Gamma(1-p)}\int_0^\infty \frac{t^{-p}\,\mathrm{e}^{-t}}{z+t}\,\mathrm{d}t,$$

which holds true for all $|\mathrm{ph}(z)| < \pi$ and $\mathrm{Re}(p) < 1$.

Lemma 2 *For all* $\nu > 0$, $a > 0$, $b > 0$ *and for all* $x > 0$ *there holds*

$$\mathscr{B}(a,b,\nu;x) = \int_0^\infty \frac{t^{\nu-1}\,\mathrm{e}^{-xt}}{at^{2\nu}+bt^{2\nu-1}+1}\,\mathrm{d}t \ge \frac{\gamma(\nu,x)\,x^{-\nu}+\Gamma(-\nu,x)\,x^\nu}{a+b+1}. \tag{3.9}$$

Proof We split the integration domain $\mathbb{R}_+ = (0,1]\cup(1,\infty)$. Accordingly we have

$$\mathscr{B}(a,b,\nu;x) \ge \int_0^1 \frac{t^{\nu-1}\,\mathrm{e}^{-xt}}{at^{2\nu}+bt^{2\nu-1}+1}\,\mathrm{d}t + \int_1^\infty \frac{t^{-\nu-1}\,\mathrm{e}^{-xt}}{a+bt^{-1}+t^{-2\nu}}\,\mathrm{d}t = B_1 + B_2.$$

Now

$$B_1 \ge \frac{1}{a+b+1}\int_0^1 t^{\nu-1}\,\mathrm{e}^{-xt}\,\mathrm{d}t = \frac{\gamma(\nu,x)}{(a+b+1)\,x^\nu},$$

while

$$B_2 \geq \frac{1}{a+b+1} \int_1^\infty t^{-\nu-1}\, \mathrm{e}^{-xt}\, \mathrm{d}t = \frac{\Gamma(-\nu, x)\; x^\nu}{(a+b+1)},$$

so the result follows. □

Now, we concentrate on the bound $\Xi_\nu(u)$, which was derived by Minakshisundaram and Szász in (3.2).

Theorem 4 *For all $x > 0$ the following lower bounds hold*

$$G_\nu(x) \geq \begin{cases} \frac{2^{-\nu-1}}{3^{1-\nu}\pi^{-\nu}} \frac{\Gamma^{2-2\nu}(\nu+1)}{\Gamma^{-2\nu}\left(\nu+\frac{1}{2}\right)} \Gamma(1-\nu)\, \exp\left\{\frac{8x\Gamma^2(\nu+1)}{3\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\right\} \Gamma\left(\nu, \frac{8x\Gamma^2(\nu+1)}{3\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\right), & \nu \in \left(-\frac{1}{2}, \frac{1}{2}\right] \\ C_1\left(\gamma(\nu, x)\, x^{-\nu} + \Gamma(-\nu, x)\, x^\nu\right), & \nu \in \left(\frac{1}{2}, \frac{3}{2}\right] \\ C_2\left(\gamma(\nu, x)\, x^{-\nu} + \Gamma(-\nu, x)\, x^\nu\right), & \nu \in \left(\frac{3}{2}, \infty\right) \end{cases},$$

where

$$C_1 = \frac{\pi^2\, \Gamma^2(2\nu+1)}{2^{2\nu+3}\left(\pi\Gamma^2\left(\nu+\frac{1}{2}\right) + 3\,\Gamma^2(\nu+1)\right) + 3\cdot 4^\nu\, \Gamma^2(\nu)\Gamma^2(2\nu+1)},$$

$$C_2 = \frac{\pi^2\, \Gamma^2(2\nu+1)}{2^{2\nu+3}\left[3\cdot 2^{2\nu-5}\left(4\,\Gamma^2(\nu+1) + \Gamma^2(\nu)\Gamma^2(2\nu+1)\right) + \pi\Gamma^2\left(\nu+\frac{1}{2}\right)\right]}.$$

Proof As $\Xi_\nu(u)$ is given by (3.2), for all $u > 0$ we have

$$\frac{1}{\eta_\nu(u)} \leq \begin{cases} \frac{3u^{2\nu+1}}{2^{2\nu}\,\Gamma^2(\nu+1)} + \frac{u^{2\nu-1}}{\pi\, 2^{2\nu-3}\,\Gamma^2\left(\nu+\frac{1}{2}\right)}, & \nu \in \left(-\frac{1}{2}, \frac{1}{2}\right] \\ \frac{u^{2\nu+1}}{2^{2\nu-2}\,\Gamma^2(\nu+1)} + \frac{3u^{2\nu-1}}{\pi\, 2^{2\nu-2}\,\Gamma^2\left(\nu+\frac{1}{2}\right)} + \frac{3\cdot 2^{2\nu}\,\Gamma^2(\nu)}{\pi^2 u^{2\nu-1}}, & \nu \in \left(\frac{1}{2}, \frac{3}{2}\right] \\ \frac{u^{2\nu+1}}{2^{2\nu-2}\,\Gamma^2(\nu+1)} + \frac{3u^{2\nu-1}}{2\pi\,\Gamma^2\left(\nu+\frac{1}{2}\right)} + \frac{3\cdot 2^{4\nu-3}\,\Gamma^2(\nu)}{\pi^2 u^{2\nu-1}}, & \nu \in \left(\frac{3}{2}, \infty\right) \end{cases}. \tag{3.10}$$

Indeed, letting $\nu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, by virtue of the inequality $a^2 + (a+b)^2 \leq 3a^2 + 2b^2$, where a, b are real, having in mind (3.7), we conclude that

$$\frac{1}{\eta_\nu(u)} \leq u(\Xi_\nu^2(u) + \Upsilon_\nu^2(u)) \leq \frac{3u^{2\nu+1}}{2^{2\nu}\,\Gamma^2(\nu+1)} + \frac{u^{2\nu-1}}{\pi\, 2^{2\nu-3}\,\Gamma^2\left(\nu+\frac{1}{2}\right)}.$$

The subsequent consideration of the another two cases of $\Upsilon_\nu(u)$, when $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right]$ and $\nu \in \left(\frac{3}{2}, \infty\right)$, confirm the another two three-term lower bound results.

The range of the Jaeger function's order parameter ν we split into three disjoint intervals like

$$\nu \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right] \cup \left(\tfrac{1}{2}, \tfrac{3}{2}\right] \cup \left(\tfrac{3}{2}, \infty\right).$$

1. $\nu \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. In this case, with the aid of Lemma 1 we have

$$G_\nu(x) \geq \int_0^\infty \frac{\mathrm{e}^{-xu^2}}{a_1 u^{2\nu-1} + a_2 u^{2\nu+1}}\,\mathrm{d}u = \frac{1}{2a_2}\int_0^\infty \frac{t^{-\nu}\,\mathrm{e}^{-xt}}{\frac{a_1}{a_2}+t}\,\mathrm{d}t = \frac{1}{2a_2}\mathscr{A}\left(\frac{a_1}{a_2}, 1-\nu; x\right),$$

where

$$a_1^{-1} = \pi 2^{2\nu-3}\,\Gamma^2\left(\nu+\tfrac{1}{2}\right), \qquad a_2^{-1} = \frac{1}{3}4^\nu\,\Gamma^2(\nu+1).$$

2. $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right]$. Now, according to the second case of (3.7) we take the first three-term lower bound for $\Upsilon_\nu(u)$ in (3.10), and we use the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for a, b, c real, which results in the estimate

$$\begin{aligned} G_\nu(x) &\geq \int_0^\infty \frac{\mathrm{e}^{-xu^2}}{b_1 u^{2\nu+1} + b_2 u^{2\nu-1} + b_3 u^{-2\nu+1}}\,\mathrm{d}u = \frac{1}{2b_3}\int_0^\infty \frac{t^{\nu-1}\,\mathrm{e}^{-xt}}{\frac{b_1}{b_3}t^{2\nu} + \frac{b_2}{b_3}t^{2\nu-1} + 1}\,\mathrm{d}t \\ &\geq \frac{1}{2(b_1+b_2+b_3)}\left(\gamma(\nu,x)x^{-\nu} + \Gamma(-\nu,x)x^\nu\right), \end{aligned}$$

where

$$b_1 = \frac{1}{4^{\nu-1}\,\Gamma^2(\nu+1)}, \qquad b_2 = \frac{3}{\pi 4^{\nu-1}\Gamma^2\left(\nu+\frac{1}{2}\right)}, \qquad b_3 = \frac{3\cdot 4^\nu\Gamma^2(\nu)}{\pi^2}.$$

By virtue of (3.9) of the Lemma 2 we prove the statement. Let us mention that the exact value of the constant C_1 was calculated with the aid of the duplication formula for the Gamma function.
Thus, the remaining third case $\nu \in \left(\frac{3}{2}, \infty\right)$ becomes obvious. □

Remark 2 The bound (3.5) for J_ν by Olenko is of the magnitude $\mathcal{O}\left(x^{-\frac{1}{2}}\right)$, while the second Landau's bound (3.4) behaviour is $\mathcal{O}\left(x^{-\frac{1}{3}}\right)$ as x approaches infinity. Therefore it can be applied by the same procedure as in the case of the Minakshisundaram–Szász bound. Also, the Ifantis–Siafarikas estimate (3.6) can be treated in the same way. Indeed, taking their bound, we have for all $u > 0$

$$\frac{1}{\eta_\nu(u)} \leq \frac{u^{2\nu+1}}{4^\nu\Gamma^2(\nu+1)}\exp\left\{-\frac{u^2}{2(\nu+1)}\right\} + u\Upsilon_\nu^2(u) \leq \frac{u^{2\nu+1}}{4^\nu\Gamma^2(\nu+1)} + u\Upsilon_\nu^2(u).$$

It remains to consider the uniform upper bounds for the Bessel function of the first kind with respect to the argument; these are von Lommel's (3.1) and the first Landau's bound (3.3). Being both argument–invariant we will denote them simply by Ξ_ν.

Theorem 5 *For all $x > 0$ we have*

$$G_\nu(x) \geq \begin{cases} C_3\,\left(\gamma(1-\nu,x)\,x^{\nu-1} + \Gamma(-\nu,x)\,x^\nu\right), & \nu \in \left(0,\frac12\right] \\ C_4\,\left(\gamma(\nu,x)\,x^{-\nu} + \Gamma(-\nu,x)\,x^\nu\right), & \nu \in \left(\frac12,\frac32\right] \\ C_5\,\left(\gamma(\nu,x)\,x^{-\nu} + \Gamma(-\nu,x)\,x^\nu\right), & \nu \in \left(\frac32,\infty\right) \end{cases}, \tag{3.11}$$

where

$$C_3 = \frac{\pi^2\,\Gamma^2(2\nu+1)}{2\pi^2\,\Gamma^2(2\nu+1)\,\Xi_\nu^2 + 4^{\nu+1}\left(\pi\,\Gamma^2\left(\nu+\frac12\right) + 4\,\Gamma^2(\nu+1)\right)},$$

$$C_4 = \frac{\pi^2\,\Gamma^2(2\nu+1)}{2\left[\pi^2\,\Gamma^2(2\nu+1)\,\Xi_\nu^2 + 3\cdot 4^\nu\left(\pi\,\Gamma^2\left(\nu+\frac12\right) + 4\,\Gamma^2(\nu+1) + \Gamma^2(\nu)\,\Gamma^2(2\nu+1)\right)\right]},$$

$$C_5 = \frac{\pi^2\,\Gamma^2(2\nu+1)}{2\left[\pi^2\,\Gamma^2(2\nu+1)\,\Xi_\nu^2 + 3\cdot 4^\nu\left(\pi\,\Gamma^2\left(\nu+\frac12\right) + 2^{2\nu-1}\,\Gamma^2(\nu+1) + 2^{2\nu-3}\,\Gamma^2(\nu)\,\Gamma^2(2\nu+1)\right)\right]},$$

$$\Xi_\nu = \begin{cases} 1, \quad \nu \in (0,1] \quad \text{and} \quad \frac{1}{\sqrt{2}}, \quad \nu \in (1,\infty) & \text{(von Lommel)} \\ \frac{b_L}{\sqrt[3]{\nu}}, \qquad \nu \in (1,\infty) & \text{(Landau)} \end{cases}$$

and $b_L = \sqrt[3]{2}\,\sup_{t\in\mathbb{R}_+} \mathrm{Ai}(t)$.

Proof Firstly, we estimate the kernel function $\eta_\nu(u)$ on the positive real half axis by bounding $J_\nu(u)$ with Ξ_ν which represents either von Lommel's (3.1) or Landau's bound (3.3).

1. Letting $\nu \in \left(0,\frac12\right]$ we get

$$\frac{1}{\eta_\nu(u)} \leq u\left(\Xi_\nu^2 + c_1 u^{2\nu-2} + c_2 u^{2\nu}\right) = u^{2\nu-1}\left(\Xi_\nu^2 u^{2-2\nu} + c_1 + c_2 u^2\right)$$

where

$$c_1 = \frac{1}{\pi\,2^{2\nu-3}\,\Gamma^2\left(\nu+\frac12\right)}, \qquad c_2 = \frac{1}{2^{2\nu-1}\,\Gamma^2(\nu+1)}.$$

Accordingly

$$G_\nu(x) \geq \frac12 \int_0^\infty \frac{t^{-\nu}\mathrm{e}^{-xt}}{\Xi_\nu^2 t^{1-\nu} + c_1 + c_2 t}\,\mathrm{d}t = \frac12\left\{\int_0^1 \frac{t^{-\nu}\mathrm{e}^{-xt}}{\Xi_\nu^2 t^{1-\nu} + c_1 + c_2 t}\,\mathrm{d}t + \int_1^\infty \frac{t^{-\nu}\mathrm{e}^{-xt}}{\Xi_\nu^2 t^{1-\nu} + c_1 + c_2 t}\,\mathrm{d}t\right\} = B_3 + B_4.$$

Since

$$B_3 \geq \frac{1}{2} \int_0^1 \frac{t^{-\nu} \mathrm{e}^{-xt}}{\left(\Xi_\nu^2 + c_2\right) t^{1-\nu} + c_1} \,\mathrm{d}t \geq \frac{\gamma(1-\nu, x)\, x^{\nu-1}}{2\left(\Xi_\nu^2 + c_1 + c_2\right)},$$

and

$$B_4 \geq \frac{1}{2} \int_1^\infty \frac{t^{-\nu-1} \mathrm{e}^{-xt}}{\left(\Xi_\nu^2 + c_2\right) + c_1 t^{-1}} \,\mathrm{d}t \geq \frac{\Gamma(-\nu, x)\, x^{\nu}}{2\left(\Xi_\nu^2 + c_1 + c_2\right)},$$

which proves the first lower bound.
2. Assume that $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right]$. Now, the appropriate upper bound is

$$\Upsilon_\nu(u) \leq \frac{1}{\Gamma(\nu+1)} \left(\frac{u}{2}\right)^\nu + \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{u}{2}\right)^{\nu-1} + \frac{\Gamma(\nu)}{\pi} \left(\frac{2}{u}\right)^\nu,$$

compare (3.7). Hence, by the procedure similar to earlier used ones, we conclude

$$\left[\eta_\nu(u)\right]^{-1} \leq u^{1-2\nu} \left(\Xi_\nu^2 u^{2\nu} + 3d_1 u^{4\nu} + 3d_2 u^{4\nu-2} + 3d_3\right),$$

where

$$d_1^{-1} = 4^\nu \Gamma^2(\nu+1), \quad d_2^{-1} = \pi 4^{\nu-1} \Gamma^2\left(\nu + \tfrac{1}{2}\right), \quad d_3^{-1} = \frac{\pi^2}{4^\nu \Gamma^2(\nu)}.$$

Thus, we have

$$\begin{aligned} G_\nu(x) &\geq \int_0^\infty \frac{u^{2\nu-1}\, \mathrm{e}^{-xu^2}\, \mathrm{d}u}{\Xi_\nu^2 u^{2\nu} + 3\left(d_1 u^{4\nu} + d_2 u^{4\nu-2} + d_3\right)} \\ &= \frac{1}{2} \int_0^\infty \frac{t^{\nu-1}\, \mathrm{e}^{-xt}\, \mathrm{d}t}{\Xi_\nu^2 t^\nu + 3\left(d_1 t^{2\nu} + d_2 t^{2\nu-1} + d_3\right)} \\ &= \frac{1}{2} \int_0^1 \frac{t^{\nu-1}\, \mathrm{e}^{-xt}\, \mathrm{d}t}{\Xi_\nu^2 t^\nu + 3\left(d_1 t^{2\nu} + d_2 t^{2\nu-1} + d_3\right)} \\ &\quad + \frac{1}{2} \int_1^\infty \frac{t^{\nu-1}\, \mathrm{e}^{-xt}\, \mathrm{d}t}{\Xi_\nu^2 t^\nu + 3\left(d_1 t^{2\nu} + d_2 t^{2\nu-1} + d_3\right)} = B_5 + B_6. \end{aligned}$$

The integral B_5 we evaluate below by applying

$$\max_{0 \leq t \leq 1} \left(\Xi_\nu^2 t^\nu + 3\left(d_1 t^{2\nu} + d_2 t^{2\nu-1} + d_3\right)\right) = \Xi_\nu^2 + 3\left(d_1 + d_2 + d_3\right) = \mathbf{M},$$

that is

$$B_5 = \frac{1}{2} \int_0^1 \frac{t^{-\nu-1}\, \mathrm{e}^{-xt}\, \mathrm{d}t}{\Xi_\nu^2 t^{-\nu} + 3\left(d_1 + d_2 t^{-1} + d_3 t^{-2\nu}\right)} \geq \frac{1}{2\mathbf{M}} \int_0^1 t^{-\nu-1}\, \mathrm{e}^{-xt}\, \mathrm{d}t = \frac{\gamma(\nu, x)}{2\mathbf{M} x^\nu},$$

while in B_6 transforming the integrand into

$$B_6 = \frac{1}{2}\int_1^\infty \frac{t^{-\nu-1}\,\mathrm{e}^{-xt}\,\mathrm{d}t}{\Xi_\nu^2 t^{-\nu} + 3\left(d_1 + d_2 t^{-1} + d_3 t^{-2\nu}\right)} \ge \frac{1}{2\mathbf{M}}\int_1^\infty t^{-\nu-1}\,\mathrm{e}^{-xt}\,\mathrm{d}t = \frac{\Gamma(-\nu, x)\,x^\nu}{2\mathbf{M}},$$

bearing in mind that also $\max_{1\le t\le\infty}\left(\Xi_\nu^2 t^{-\nu} + 3(d_1 + d_2 t^{-1} + d_3 t^{-2\nu})\right) = \mathbf{M}$.
3. $\nu \in \left(\frac{3}{2}, \infty\right)$. In this case the upper bound becomes

$$\Upsilon_\nu(u) \le \frac{1}{\Gamma(\nu+1)}\left(\frac{u}{2}\right)^\nu + \frac{u^{\nu-1}}{\sqrt{2\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} + \frac{2^{2\nu-\frac{3}{2}}\Gamma(\nu)}{\pi u^\nu},$$

see (3.7). However, being the bound of the same fashion as in the previous case we get

$$\left[\eta_\nu(u)\right]^{-1} \le u^{1-2\nu}(\Xi_\nu^2 u^{2\nu} + 3e_1 u^{4\nu} + 3e_2 u^{4\nu-2} + 3e_3),$$

with the coefficients

$$e_1^{-1} = 4^\nu\Gamma^2(\nu+1),\quad e_2^{-1} = 2\pi\,\Gamma^2\left(\nu+\tfrac{1}{2}\right)\quad\text{and}\quad e_3^{-1} = \frac{\pi^2}{2^{4\nu-3}\,\Gamma^2(\nu)},$$

which yields C_5 as well. □

It is important to mention here that by using the bilateral bounding inequalities for the lower incomplete gamma $\gamma(p^{-1}, x^p),\ p > 0$ by Alzer [28] who specified some earlier results by Gautschi [29] it is possible to give further specifications regarding lower bounds for the real parameter Jaeger integral $G_\nu(x)$ for $\nu > 0$. In this purpose we list Alzer's result [28, p. 772, Theorem 1]: *Let $p \ne 1$ be a positive real number, and let $\beta = \beta(p)$ be given by*

$$\alpha = 1,\qquad \beta = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{-p}\quad\text{if}\quad p \in (0,1), \tag{3.12}$$

and

$$\alpha = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{-p},\qquad \beta = 1\quad\text{if}\quad p \in (1,\infty).$$

Then for all positive $x > 0$ we have

$$\left(1-\mathrm{e}^{-\beta x^p}\right)^{\frac{1}{p}} < \frac{\gamma\left(\frac{1}{p}, x^p\right)}{\Gamma\left(\frac{1}{p}\right)} < \left(1-\mathrm{e}^{-\alpha x^p}\right)^{\frac{1}{p}}. \tag{3.13}$$

Now, certain corollaries can be inferred of Theorem 4 and Theorem 5 achieving new lower bounds for both Minakshisundaram and Szász (3.2), and von Lommel (3.1) or

Landau's (3.3) upper bound cases by means of the previously recalled Alzer's contribution (3.12). Bearing in mind that both Theorem 4 and Theorem 5 contain bounds of lower and upper incomplete gamma functions, that is, either $\gamma(1-\nu, x)$, $\gamma(\nu, x)$ or $\Gamma(-\nu, x)$, say, the bounds in (3.13) unfortunately cannot be mutually applicable being the parameter od the latter negative.

Thus, for example putting $\nu = \frac{1}{p}$ in (3.12) we obtain the following result.

Corollary 1 (to Theorem 4) *For all* $x > 0$ *the following lower bounds hold*

$$G_\nu(x^{\frac{1}{\nu}}) \geq \begin{cases} \dfrac{2^{-\nu-1}}{3^{1-\nu}\pi^{-\nu}} \dfrac{\Gamma^{2-2\nu}(\nu+1)}{\Gamma^{-2\nu}\left(\nu+\frac{1}{2}\right)} \Gamma(1-\nu) \exp\left\{\dfrac{8x^{\frac{1}{\nu}}\Gamma^2(\nu+1)}{3\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\right\} \Gamma\left(\nu, \dfrac{8x^{\frac{1}{\nu}}\Gamma^2(\nu+1)}{3\pi\Gamma^2\left(\nu+\frac{1}{2}\right)}\right), & \nu \in \left(-\frac{1}{2}, \frac{1}{2}\right] \\ C_1\Big(\Gamma(\nu)\, x^{-1} \left(1-\mathrm{e}^{-\beta x^{\frac{1}{\nu}}}\right)^\nu + x\,\Gamma\left(-\nu, x^{\frac{1}{\nu}}\right)\Big), & \nu \in \left(\frac{1}{2}, \frac{3}{2}\right] \\ C_2\Big(\Gamma(\nu)\, x^{-1} \left(1-\mathrm{e}^{-\beta x^{\frac{1}{\nu}}}\right)^\nu + x\,\Gamma\left(-\nu, x^{\frac{1}{\nu}}\right)\Big), & \nu \in \left(\frac{3}{2}, \infty\right) \end{cases}, \tag{3.14}$$

where the constants C_1, C_2 *exposed in* Theorem 4 *arrive unaltered and*

$$\beta = \beta(\nu) = \begin{cases} \left(\Gamma\,(1+\nu)\right)^{-\frac{1}{\nu}}, & \nu \in (0, 1) \\ 1, & \nu \in (1, \infty) \end{cases}.$$

Moreover, we have the next result.

Corollary 2 (to Theorem 5) *For all* $x > 0$ *we have*

$$G_\nu(x) \geq \begin{cases} C_3\left(\Gamma(1-\nu)\, x^{-1} \left(1-\mathrm{e}^{-\beta x^{\frac{1}{1-\nu}}}\right)^{1-\nu} + x^{\frac{\nu}{1-\nu}}\,\Gamma\left(-\nu, x^{\frac{1}{1-\nu}}\right)\right), & \nu \in \left(0, \frac{1}{2}\right] \\ C_4\left(\Gamma(\nu)\, x^{-1} \left(1-\mathrm{e}^{-\beta x^{\frac{1}{\nu}}}\right)^\nu + x\,\Gamma\left(-\nu, x^{\frac{1}{\nu}}\right)\right), & \nu \in \left(\frac{1}{2}, \frac{3}{2}\right] \\ C_5\left(\Gamma(\nu)\, x^{-1} \left(1-\mathrm{e}^{-\beta x^{\frac{1}{\nu}}}\right)^\nu + x\,\Gamma\left(-\nu, x^{\frac{1}{\nu}}\right)\right), & \nu \in \left(\frac{3}{2}, \infty\right) \end{cases}, \tag{3.15}$$

where C_3, C_4 *and* C_5 *remain the same, being* b_L *the Laundau's first constant.*

Acknowledgments The authors are indebted to both unknown referees and to the editor for their constructive comments and suggestions, which substantially encompassed the article complementing the study.

References

1. J.C. Jaeger, Proc. R. Soc. Edinb. A **61**, 223 (1942)
2. W.R.C. Phillips, P.J. Mahon, Proc. R. Soc. A **467**, 3570 (2011)
3. Á. Baricz, T.K. Pogány, Acta Polytech. Hung. **10**, 53 (2013)
4. Á. Baricz, T.K. Pogány, Math. Inequal. Appl. **17**, 989 (2014)
5. L.P. Smith, J. Appl. Phys. **8**, 441 (1937)
6. J.C. Jaeger, Proc. R. Soc. Edinb. A **61**, 223 (1941)
7. H.S. Carlslaw, J.C. Jaeger, *Conduction of Heat in Solids*, 2nd edn. (Oxford University Press, Oxford, 1959)
8. J.W. Nicholson, Proc. R. Soc. Lond. A **100**, 226 (1921)
9. A.S. Chessin, Comptes Rendus **135**, 678 (1902)

10. A.S. Chessin, Comptes Rendus **136**, 1124 (1903)
11. Á. Baricz, D. Jankov, T.K. Pogány, Integr. Transf. Spec. Funct. **23**, 529 (2012)
12. E. Grosswald, Ann. Probab. **4**, 680 (1976)
13. M.E.H. Ismail, Ann. Probab. **5**, 582 (1977)
14. Y. Hamana, Osaka J. Math. **49**, 853 (2012)
15. Y. Hamana, H. Matsumoto, T. Shirai, arXiv:1302.5154v1 (2013)
16. E.C.J. von Lommel, Abh. der math. phys. Classe der k. b. Akad. der Wiss. (München) 15 529 (1884–1886)
17. G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1922)
18. S. Minakshisundaram, O. Szász, Trans. Am. Math. Soc. **61**, 36 (1947)
19. L. Landau, in *Proceedings of the Symposium on Mathematical Physics and Quantum Field Theory*, ed. by H. Warchall (Berkeley, CA: June 11–13, 1999), 147 (2000); Electronic J. Differential Equations 4 (2002), Southwest Texas State University, San Marcos, TX
20. I. Krasikov, J. Appl. Anal. **12**, 83 (2006)
21. A.Y. Olenko, Integral Transforms Spec. Funct. **17**, 455 (2006)
22. E.K. Ifantis, P.D. Siafarikas, J. Math. Anal. Appl. **147**, 214 (1990)
23. Á. Baricz, P.L. Butzer, T.K. Pogány, in *Analytic Number Theory, Approximation Theory, and Special Functions—In Honor of Hari M. Srivastava*, vol. 775, ed. by T. Rassias, G. V. Milovanović (Springer, New York, 2014)
24. P. Cerone, in *Advances in Inequalities for Special Functions*, vol. 1, ed. by P. Cerone, S. Dragomir (Nova Science Publishers, New York, 2008)
25. H.M. Srivastava, T.K. Pogány, Russian J. Math. Phys. **14**, 194 (2007)
26. Á. Baricz, D. Jankov, T.K. Pogány, Proc. Am. Math. Soc. **140**, 951 (2012)
27. F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (eds.), *NIST Handbook of Mathematical Functions* (Cambridge University Press, Cambridge, 2010)
28. H. Alzer, Math. Comput. **66**, 771 (1997)
29. W. Gautschi, J. Math. Phys. **38**, 77 (1959)